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Calculating Entropy of Plane Symmetry Black Hole via Generalized Uncertainty Relation

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Abstract The generalized uncertainty relation is introduced to calculate entropy of the black hole. By using quantum statistical method, we directly obtain the partition function of Bose and Fermi field on the background of the plane symmetry black hole. Then we calculate the entropy of Bose and Fermi field on the background of black hole near the horizon of the black hole. In our calculation, we need not introduce cutoff. There are not the left out term and the divergent logarithmic term in the original brick-wall method. And it is obtained that the entropy of the black hole is proportional to the area of the horizon. The inherent contact between the entropy of black hole and the area of horizon is opened out. Further it is shown the entropy of black hole is entropy of quantum state on the surface of horizon. The black hole's entropy is the intrinsic property of the black hole. The entropy is a quantum effect.

Keywords Brick-wall method · Entropy of black hole · Quantum statistics · Generalized uncertainty relation

1 Introduction

Hawking [1] interpreted the quantum effect of black hole as event horizon emits thermal radiation spectral particles. This was a milestone in black hole physics. The discovery not only solved the contradiction in black hole thermodynamics but also indicated the inherent contact among quantum dynamics, thermodynamics and gravity. Discussing thermal properties of various black holes becomes an important subject in black hole physics [2]. There have been a number of efforts in the past several years aimed at the entropy of black hole. The statistical origin of black hole has been probed and many methods of calculating entropy have emerged [3–8]. One frequently used method is the brick-wall method advanced by 't Hooft [6]. This method is used to study the statistical properties of the scalar field

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and Dirac fields in various black holes [9–13] and it is found that the black-hole entropy is proportional to the area of its horizon. To solve the problem that state density diverges at horizon, we need introduce cutoff. But introducing cutoff let us feel unnatural. Lately, Chang pointed that considering the gravity correction to uncertainty relation would lead to the change of general state density equation. Furthermore, if we generalize the results of Ref. [14] to curved spacetime quantum field theory, we will solve the divergent problem in the original brick-wall method [15–17].

By using the method of quantum statistics, we derive directly the partition functions of Bose and Fermi field in the plane symmetry black hole and avoid the difficult to solve various particle wave equations. The integral expression of black plane entropy is obtained. However, using equation of state density motivated by generalized uncertainty relation calculate the statistical entropy corresponds to the horizon of black plane, we derive that the entropy of the black hole is proportional to the area of the horizon. In our calculation, we need not introduce cutoff. There are not the left out term and the divergent logarithmic term in the original brick-wall method. Result shows that a black hole's entropy is the property of horizon as a null hypersurface. The black hole's entropy is the intrinsic property of the black hole. Hence, we provide a method to interpret how entropy is taken as a measure of number of microstate. Obviously, the method in this paper is a more useful tool to solve quantum gravity problems and to look for the relation between statistical dynamics and thermodynamics. In this paper, we take the simplest function form of temperature ($\hbar = c = G = K_B = 1$).

2 Plane Symmetry Spacetime

The linear element in plane symmetry spacetime is given by [18]:

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + C(r)(dx^2 + dy^2), \tag{1}$$

where

$$B(r) = -\frac{4\pi M}{N\alpha^N} r^{1-N} + \frac{6\alpha^2}{N(2N-1)} r^N + \frac{2Q^2}{N\alpha^{2N}} r^{-N}, \qquad C(r) = (\alpha r)^N.$$

When $1/2 < N < 2$, the location of horizon of black hole satisfies $B(r) = 0$, namely

$$\frac{3\alpha^2}{(2N-1)} r^{2N} - \frac{2\pi M}{\alpha^N} r + \frac{Q^2}{\alpha^{2N}} = 0. \tag{2}$$

The radiation temperature of the black hole is:

$$\begin{aligned} T_H = \beta_0^{-1} &= \frac{1}{2\pi}\left(-\frac{2\pi M(1-N)}{N\alpha^N} r_H^{-N} + \frac{3\alpha^2}{(2N-1)} r_H^{N-1} - \frac{Q^2}{\alpha^{2N}} r_H^{-N-1}\right) \\ &= \frac{1}{2\pi}\frac{B'(r_H)}{2}. \end{aligned} \tag{3}$$

Area of a horizon of a black plane corresponds to unit xoy plane is:

$$A_H = (\alpha r_H)^{2N}. \tag{4}$$

3 Bosonic Entropy

Generalized uncertainty relation [14, 19–21]

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2. \tag{5}$$

In dVd^3P phase volume, the number of quantum states is given by

$$\frac{dVd^3p}{(2\pi\hbar)^3(1+\lambda p^2)^3}, \tag{6}$$

Here λ is the measure of Planck length.

In the view of Ref. [22], the natural radiation temperature near the horizon of black hole is as follows:

$$T = \frac{T_H}{\sqrt{f}}, \tag{7}$$

where $\sqrt{f} = \sqrt{B(r)}$, is red-shift factor.

For bosonic gas, we calculate the partition function of the system as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}). \tag{8}$$

In unit volume, from (6), the number of quantum states with radiation frequency of particles less than or equal to ν is given by

$$g(\nu) = j\frac{4\pi p^3}{3(2\pi\hbar)^3(1+\lambda p^2)^3} = j\frac{4\pi\nu^3}{3(1+\lambda 4\pi^2\nu^2)^3}, \tag{9}$$

where j is the spinning degeneracy of particles. Since in space-time (1), a hypersurface corresponding unit xoy plane at arbitrary point r is $\alpha^{2N}r^{2N}$, the partition function of the system at the lamella with arbitrary thickness outside the horizon of black plane is

$$\begin{aligned}
\ln Z &= -\alpha^{2N}\int r^{2N}\frac{dr}{\sqrt{f}}\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}) \\
&= -\alpha^{2N}\int r^{2N}\frac{dr}{\sqrt{f}}\int_0^\infty dg(\nu)\ln(1 - e^{-\beta h\nu}) \\
&= j\alpha^{2N}\int r^{2N}\frac{dr}{\sqrt{f}}\int_0^\infty \frac{4\pi\beta h}{3(1+\lambda 4\pi^2\nu^2)^3(e^{\beta h\nu}-1)}\nu^3 d\nu \\
&= j\beta_0\alpha^{2N}\int r^{2N}dr\int_0^\infty \frac{4\pi h}{3(1+4\pi^2\nu^2)^3(e^{\beta h\nu}-1)}\nu^3 d\nu,
\end{aligned} \tag{10}$$

where $\beta=\beta_0\sqrt{-g_{tt}}$. According to the relation between free energy and partition function, we have

$$F=-\frac{1}{\beta_0}\ln Z=-j\alpha^{2N}\int r^{2N}\frac{dr}{\sqrt{f}}\int_0^\infty\frac{4\pi h\sqrt{-g_{tt}}}{3(1+\lambda 4\pi^2\nu^2)^3(e^{\beta h\nu}-1)}\nu^3 d\nu. \tag{11}$$

So the entropy of the system is

$$\begin{aligned}S_b&=\beta_0^2\frac{\partial F}{\partial\beta_0}=j\beta_0\alpha^{2N}\int r^{2N}\frac{dr}{\sqrt{f}}\int_0^\infty\frac{4\pi\beta\nu h^2\sqrt{-g_{tt}}e^{\beta h\nu}}{3(1+\lambda 4\pi^2\nu^2)^3(e^{\beta h\nu}-1)^2}\nu^3 d\nu\\&=j\frac{\alpha^{2N}}{6\pi^2\beta_0^3}\int\frac{r^{2N}dr}{(-g_{tt}^{3/2}\sqrt{f}}\int_0^\infty\frac{e^x x^4 dx}{(1+\lambda\frac{x^2}{\beta_0^2(-g_{tt})})(e^x-1)^2},\end{aligned} \tag{12}$$

where $x=\beta h\nu$, suppose

$$\begin{aligned}I_1(g_{tt})&=\int_0^\infty\frac{e^x x^4 dx}{(1+\lambda\frac{x^2}{\beta_0^2(-g_{tt})})(e^x-1)^2}\\&\approx\int_0^\infty\frac{(x^2+x^3)dx}{(1+\lambda\frac{x^2}{\beta_0^2(-g_{tt})})^3}=\frac{\pi}{16}\beta_0^3\left(\frac{-g_{tt}}{\lambda}\right)^{3/2}+\frac{1}{4}\beta_0^4\left(\frac{-g_{tt}}{\lambda}\right)^2.\end{aligned} \tag{13}$$

In (12), we integrate and take the integral region $[r_H, r_H+\varepsilon]$. Then substituting (13) into (12), we obtain

$$\begin{aligned}S_b&=j\frac{\alpha^{2N}}{6\pi^2\beta_0^3}\int_{r_H}^{r_H+\varepsilon}\frac{r^{2N}dr}{g_{tt}^2}\left[\frac{\pi}{16}\beta_0^3\left(\frac{-g_{tt}}{\lambda}\right)^{3/2}+\frac{1}{4}\beta_0^4\left(\frac{-g_{tt}}{\lambda}\right)^2\right]\\&=j\frac{\alpha^{2N}r_H^{2N}}{6\pi^2}\left[\frac{\pi}{16\lambda^{3/2}}\sqrt{\frac{2\varepsilon}{\kappa}}+\frac{1}{4}\beta_0\frac{\varepsilon}{\lambda^2}\right].\end{aligned} \tag{14}$$

We are only interested in the contribution from the vicinity near the horizon. From generalized uncertainty relation (5), we derive that the minimal uncertainty degree is $2\sqrt{\lambda}$ under Planck scale. Hence, taking as the minimal length of linear element of pure spacetime, $2\sqrt{\lambda}$ have the following form.

$$2\sqrt{\lambda}=\int_{r_H}^{r_H+\varepsilon}\frac{dr}{\sqrt{f}}\approx\int_{r_H}^{r_H+\varepsilon}\frac{dr}{\sqrt{2\kappa(r-r_H)}}=\sqrt{\frac{2\varepsilon}{\kappa}}, \tag{15}$$

where κ is the surface gravity at the horizon of black hole and it is identified as $\kappa=2\pi\beta_0^{-1}$. Thus we naturally derive the expression of entropy

$$S_b=j\frac{3\alpha^{2N}r_H^{2N}}{16\lambda\pi}=j\frac{3A_H}{16\lambda\pi}, \tag{16}$$

where $A_H = \alpha^2 r_H^2$, is the area of outer horizon of black plane corresponds to unit xoy plane. Using the new equation of state density motivated by the generalized uncertainty relation calculate the statistical entropy corresponds to the horizon of black plane, we obtain that the entropy of spacetime is proportional to the area of the black hole's horizon. In our calculation, we need not introduce cutoff. The value of entropy has nothing to do with the radiation field outside the horizon. And the horizon only has the property of the one-dimensional membrane in three-dimensional space. So the entropy is property of this one-dimensional membrane. It should be entropy of the black plane.

4 Fermi Entropy

For Fermi gas, the partition function is as follows:

$$\ln Z = \sum_i g_i \ln(1 + e^{-\beta\varepsilon_i}). \tag{17}$$

From (9), we have

$$\begin{aligned} S_f &= \beta_0^2 \frac{\partial F}{\partial \beta_0} = j\beta_0 \int A(r)dr \int_0^\infty \frac{4\pi\beta\nu e^{\beta h\nu} h^2}{3(1+\lambda 4\pi^2\nu^2)^3 (e^{\beta h\nu}+1)^2} \nu^3 d\nu \\ &= j\frac{1}{6\pi^2\beta_0^3} \int \frac{A(r)dr}{f^2} \int_0^\infty \frac{e^x x^4 dx}{(1+\lambda\frac{x^2}{\beta_0^2 f})^3 (e^x+1)^2}, \end{aligned} \tag{18}$$

$$\begin{aligned} I_2(f) &= \int_0^\infty \frac{e^x x^4 dx}{(1+\lambda\frac{x^2}{\beta_0^2 g_{tt}})^3 (e^x+1)^2} \\ &= \int_0^\infty \frac{e^x x^4 dx}{(1+\mu x^2)^3 (e^x+1)^2} = \frac{1}{2}\frac{\partial^2}{\partial \mu^2} \int_0^\infty \frac{e^x dx}{(1+\mu x^2)(e^x+1)^2} \\ &= -\frac{\partial^2}{\partial \mu^2} \int_0^\infty \frac{\mu x dx}{(e^x+1)(1+\mu x^2)^2} \approx -\frac{\partial^2}{\partial^2 \mu} \int_0^\infty \frac{\mu x dx}{(x+2)(1+\mu x^2)} \\ &= -\frac{\partial^2}{\partial \mu^2} \int_0^\infty \left[\frac{\mu}{(1+\mu x^2)} - \frac{2\mu}{(x+2)(1+\mu x^2)}\right] dx \\ &= -\frac{\partial^2}{\partial \mu^2} \left[\frac{\sqrt{\mu}}{2}\pi - \frac{\mu}{4\mu+1}(\ln(2+x)^2 - \ln(1+\mu x^2) - 4\sqrt{\mu}\,\mathrm{arctg}\, x\sqrt{\mu})\right]_0^\infty \\ &\approx \frac{1}{4}\mu^{-2} = \frac{1}{4}\beta_0^4 \left(\frac{f}{\lambda}\right)^2. \end{aligned} \tag{19}$$

So the entropy of the black hole corresponds to Fermi field is

$$S_f = \omega \frac{\alpha^{2N}}{6\pi^2 \beta_0^3} \int_{r_H}^{r_H+\varepsilon} \frac{r^{2N} dr}{4g_{tt}^2} \beta_0^4 \left(\frac{g_{tt}}{\lambda}\right)^2 = \omega \frac{\alpha^{2N} r_H^{2N}}{6\pi^2} \frac{1}{4} \beta_0 \frac{\varepsilon}{\lambda^2} = \omega \frac{\alpha^{2N} r_H^{2N}}{6\pi \lambda} \tag{20}$$

where ω is the spinning degeneracy of fermions.

5 Conclusion

As early as 1992, Li and Liu phenomenally proposed the state equations motivated by gravity and gave the state equations of the thermal radiation filed near the horizon of black hole [23]. Using the Li–Liu equation, Wang calculated the entropy of a black hole and obtained that the entropy of the black hole is proportional to the area of the horizon [24]. However, in his calculation the left out term and the divergent logarithmic term in the original brick-wall method don't exist.

Based on the above analysis, we calculate the entropy of the plane symmetry black hole by using the new equation of state density motivated by the generalized uncertainty relation. The divergence appearing in the brick wall model is removed, without any cutoff. It is derived that the entropy is proportional to the area of the horizon. We start with different consideration, but obtain the same conclusion. So there exist the inherent contact between the Li–Liu equation and generalized uncertainty relation. And this inherent contact is a subject of theoretic physics that we should study.

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